

# ORTHOGONAL POLYNOMIALS AND OPERATOR ORDERINGS

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**ABSTRACT.** An alternative and combinatorial proof is given for a connection between a system of Hahn polynomials and identities for symmetric elements in the Heisenberg algebra, which was first observed by Bender, Mead, and Pinsky [Phys. Rev. Lett. 56 (1986), J. Math. Phys. 28, 509 (1987)] and proved by Koornwinder [J. Phys. Phys. 30(4), 1989]. In the same vein two results announced by Bender and Dunne [J. Math. Phys. 29 (8), 1988] connecting a special one-parameter class of Hermitian operator orderings and the continuous Hahn polynomials are also proved.

## 1. INTRODUCTION

The Meixner-Pollaczek polynomials are defined by

$$P_n^{(a)}(x; \phi) = \frac{(2a)_n}{n!} e^{in\phi} {}_2F_1(-n, a + ix; 2a; 1 - e^{-2i\phi}).$$

Consider the following special Meixner-Pollaczek polynomial:

$$S_n(x) = n! P_n^{(1/2)}\left(\frac{1}{2}x; \frac{1}{2}\pi\right) = i^n n! \sum_{k=0}^n \frac{(-1)^k}{k!} \binom{n}{k} \prod_{j=0}^{k-1} (ix + 1 + 2j), \quad (1)$$

which turns out to be the orthogonal polynomial of degree  $n$  on  $\mathbb{R}$  with respect to the weight function  $x \mapsto 1/\cosh(\pi x/2)$ . Clearly we have

$$S_{n+1}(x) = xS_n(x) - n^2 S_{n-1}(x)$$

with  $S_0(x) = 1$ . The first values of these polynomials are as follows:

$$S_1(x) = x, \quad S_2(x) = x^2 - 1, \quad S_3(x) = x^3 - 5x, \quad S_4(x) = x^4 - 14x^2 + 9.$$

Let  $\mathfrak{S}_n$  be the set of permutations on  $\{1, 2, \dots, n\}$ . For any  $\sigma \in \mathfrak{S}_n$  let  $\text{cyco } \sigma$  be the number of cycles in  $\sigma$  of odd length. Then it is easy to see that the polynomial  $S_n(x)$  has the following combinatorial interpretation:

$$S_n(x) = (-i)^n \sum_{\sigma \in \mathfrak{S}_n} (ix)^{\text{cyco } \sigma}.$$

It is interesting to note that the corresponding moment is the secant number  $E_{2n}$  defined by

$$\sum_{n \geq 0} E_{2n} \frac{x^{2n}}{(2n)!} = \frac{1}{\cos x}.$$

If

$$[q, p] := qp - pq = i, \quad (2)$$

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and  $T_n$  is the sum of all possible terms containing  $n$  factors of  $p$  and  $n$  factors of  $q$ , then Bender, Mead and Pinsky [1, 2] first observed and Koornwinder [6] proved the following result

$$T_n = \frac{(2n-1)!!}{n!} S_n(T_1). \quad (3)$$

For example, we have  $T_1 = pq + qp$  and  $T_2 = T_1^2 + p^2q^2 + q^2p^2$ . It follows from (2) that  $[q, p]^2 = (qp)^2 - qp^2q - pq^2p + (pq)^2 = -1$ . Hence  $T_1^2 - 1 = 2((qp)^2 + (pq)^2)$  and

$$p^2q^2 + q^2p^2 + 1 = p(pq + i)q + q(qp - i)p = (pq)^2 + (qp)^2 = \frac{1}{2}(T_1^2 - 1). \quad (4)$$

Finally we get  $T_2 = (T_1^2 - 1) + (p^2q^2 + q^2p^2 + 1) = \frac{3}{2}S_2(T_1)$ .

To prove (3), Koornwinder [6] made use of the connection between Laguerre polynomials and Meixner-Pollaczek polynomials, the Rodrigues formula for Laguerre polynomials, an operational formula involving Meixner-Pollaczek polynomials, and the Schrödinger model for the irreducible unitary representations of the three-dimensional Heisenberg group. To the best knowledge of the authors Koornwinder's proof is the only published one for (3). In this paper we shall give an elementary proof of (3) using the rook placement interpretation of the normal ordering of two non commutative operators. See [7], and also [8, 4] for two recent papers on this theory.

On the other hand, Bender and Dunne [3] discussed a correspondence between polynomials and rules for operator orderings. More precisely, given two operators  $q$  and  $p$  satisfying (2), they consider the possible operator orderings  $O$  as a sum

$$O(q^n p^n) = \frac{\sum_{k=0}^n a_k^{(n)} q^k p^n q^{n-k}}{\sum_{k=0}^n a_k^{(n)}},$$

where the coefficients  $a_k^{(n)}$  may be chosen arbitrarily with  $a_0^{(0)} = a_0^{(1)} = a_1^{(1)} = 1$ . Hence  $O(q^0 p^0) = 1$  and  $O(q^1 p^1) = \frac{1}{2}(qp + pq)$ . They pointed out that with every operator correspondence rule  $O$  one can associate a class of polynomials  $p_n(x)$  defined by

$$O(q^n p^n) = p_n[O(qp)]. \quad (5)$$

If  $a_k^{(n)} = \binom{n}{k}$ , their result is equivalent to

$$O(q^n p^n) := \sum_{k=0}^n \binom{n}{k} q^k p^n q^{n-k} = S_n(T_1). \quad (6)$$

Their polynomial is actually equal to  $2^{-n}S_n(2x)$ . For example, we have

$$O(q^2 p^2) = q^2 p^2 + 2qp^2q + p^2 q^2 = \frac{T_1^2 - 3}{2} + 2\frac{T_1 - i}{2} \cdot \frac{T_1 + i}{2} = T_1^2 - 1.$$

If

$$a_k^{(n)} = \binom{n+l}{k} \binom{n+l}{k+l} \binom{n+l}{l}^{-1}, \quad (7)$$

where  $l$  is an arbitrary parameter, Bender and Dunne [3] observed that the corresponding polynomials belong to the large class of continuous Hahn polynomials. An explicit formula for the  $n$ th polynomial is

$$P_n(x) = i^n \binom{2n+2l}{n}^{-1} \frac{\Gamma(n+l+1)}{\Gamma(l+1)} {}_3F_2\left(-n, n+2l+1, \frac{1}{2} + ix; 1, l+1; 1\right).$$

Bender and Dunne [3] announced the following result

$$O(q^n p^n) := \frac{\sum_{k=0}^n a_k^{(n)} q^k p^n q^{n-k}}{\sum_{k=0}^n a_k^{(n)}} = P_n(T_1/2). \quad (8)$$

Note that the denominator has a closed formula

$$D_n := \sum_{k=0}^n a_k^{(n)} = \binom{n+l}{l}^{-1} \sum_{k=0}^n \binom{n+l}{k} \binom{n+l}{n-k} = \binom{n+l}{l}^{-1} \binom{2n+2l}{n}.$$

For example, by (4) we have

$$\begin{aligned} O(q^2 p^2) &= \frac{1}{2} \frac{1+l}{3+2l} (p^2 q^2 + 2 \frac{2+l}{1+l} q p^2 q + q^2 p^2) \\ &= \frac{1}{2} \frac{1+l}{3+2l} \left( \frac{T_1^2 - 3}{2} + \frac{2+l}{1+l} \frac{(T_1 + i)(T_1 - i)}{2} \right) \\ &= \frac{T_1^2}{4} - \frac{1}{4} \frac{1+2l}{3+2l} = P_2(T_1/2). \end{aligned}$$

Since (6) and (8) were announced without proof, we shall provide a proof similar to that of (3).

We shall first recall briefly the rook theory of normal ordering in Section 2 and then prove (3), (6) and (8) in Sections 3, 4 and 5, respectively.

## 2. ROOK PLACEMENTS AND THE NORMAL ORDERING PROBLEM

Let  $D$  and  $U$  be two operators satisfying the commutation relation  $[D, U] = 1$ . Then the algebra generated by  $D$  and  $U$  is the Weyl algebra. Each element of this algebra, identified as a word  $w$  on the alphabet  $\{D, U\}$ , can be uniquely written in the normally ordered form as

$$w = \sum_{r,s} c_{r,s} U^r D^s.$$

The coefficients  $c_{r,s}$  can be computed using the rook theory. The reader is referred to [8, 4] for more details. Given a word  $w$  with  $n$  letters  $U$ 's and  $m$  letters  $D$ 's, we draw a lattice path from  $(n, 0)$  to  $(0, m)$  as follows: read the word  $w$  from left to right and draw a unit line to the right (resp. down) if the letter is  $D$  (resp.  $U$ ). This lattice path outlines a Ferrers diagram  $B_w$  as illustrated in Figure 1.

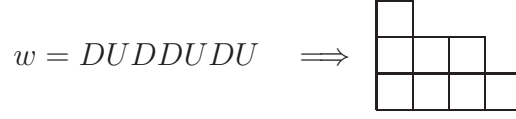


Figure 1. Correspondence between words and Ferrers diagrams

The commutation rule  $DU = UD + 1$  implies that the normal writing of  $w$  amounts to replacing successively each  $DU$  by  $UD$  or 1, this procedure amounts to deleting each *up-right-most* corner of the Ferrers board or deleting it with its row and column. Let  $r_k(B)$  be the number of placing  $k$  ( $k \geq 0$ ) non-attacking rooks on the Ferrers board  $B_w$ . It is known (see [7, 8]) that

$$w = \sum_k r_k(B_w) U^{n-k} D^{m-k}. \quad (9)$$

Now, it follows from [8, Theorem 5.1] that

$$\sum_{B \subseteq [n] \times [n]} r_k(B) = \frac{(2n)!}{2^k k! (n-k)! (n-k)!}, \quad (10)$$

where the sum is over all the Ferrers diagrams contained in the square  $[n] \times [n]$  and outlined by the lattice paths starting from  $(0, n)$  and ending at  $(n, 0)$ . Let  $T_n(D, U)$  be the sum of all the words with  $n$  letters  $D$  and  $n$  letters  $U$ . We derive from (9) and (10) that

$$T_n(D, U) = \sum_{k=0}^n \frac{(2n)!}{2^k k! (n-k)! (n-k)!} U^{n-k} D^{n-k}. \quad (11)$$

On the other hand, let  $x := DU + UD = 2UD + 1$ , then

$$UD = \frac{x-1}{2}. \quad (12)$$

It is also folklore (see [5, p. 310]) that

$$U^n D^n = \sum_{k=0}^n s(n, k) (UD)^k, \quad (13)$$

where  $s(n, k)$  is the *Stirling number of the first kind*. Since

$$\sum_{j=0}^n s(n, j) t^j = t(t-1) \cdots (t-n+1),$$

we can rewrite (13) as

$$U^n D^n = \prod_{j=1}^n (UD - j + 1). \quad (14)$$

### 3. PROOF OF EQUATION (3)

If we set  $D = -iq$  and  $U = p$ , then equation (2) becomes  $[D, U] = 1$  and the algebra generated by  $D$  and  $U$  is the Weyl algebra. Substituting (14) and (12) into (11), we obtain, by replacing  $k$  by  $n - k$ ,

$$T_n(D, U) = (2n - 1)!! \sum_{k=0}^n \frac{(-1)^k}{k!} \binom{n}{k} \prod_{j=0}^{k-1} (-x + 2j + 1) = (-i)^n \frac{(2n - 1)!!}{n!} S_n(ix).$$

Since  $T_n(D, U) = (-i)^n T_n(q, p)$ , letting  $T_1 = pq + qp = ix$ , we derive

$$T_n(q, p) = \frac{(2n - 1)!!}{n!} S_n(T_1),$$

which is exactly (3).

### 4. PROOF OF EQUATION (6)

For the commutation relation  $DU - UD = 1$  and for  $n \geq k \geq 0$ , it is readily seen, by induction on  $k$ , that

$$D^k U^n = \sum_{j=0}^k \binom{k}{j} n^j U^{n-j} D^{k-j}, \quad (15)$$

where  $n^j = n(n - 1) \dots (n - j + 1) = n! / (n - j)!$ , then

$$\begin{aligned} O(D^n U^n) &= \sum_{k=0}^n \binom{n}{k} \sum_{j=0}^k \binom{k}{j} n^j U^{n-j} D^{n-j} \\ &= \sum_{j=0}^n \binom{n}{j} 2^{n-j} n^j U^{n-j} D^{n-j}. \end{aligned}$$

Substituting (14) and (12) in the last sum, we obtain, by replacing  $n - j$  by  $k$ ,

$$O(D^n U^n) = \sum_{k=0}^n (-1)^k \binom{n}{k} n^{n-k} \prod_{j=0}^{k-1} (-x + 2j + 1) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{n!}{k!} \prod_{j=0}^{k-1} (-x + 2j + 1).$$

As  $T_1 = qp + pq = ix$  we derive  $O(q^n p^n) = i^n O(D^n U^n) = S_n(ix)$ .

## 5. PROOF OF EQUATION (8)

By definition and (15), we have

$$\begin{aligned}
O(D^n U^n) &= \binom{2n+2l}{n}^{-1} \sum_{k=0}^n \binom{n+l}{k} \binom{n+l}{k+l} D^k U^n D^{n-k} \\
&= \binom{2n+2l}{n}^{-1} \sum_{k=0}^n \binom{n+l}{k} \binom{n+l}{k+l} \sum_{j=0}^k \binom{k}{j} n^j U^{n-j} D^{n-j} \\
&= \binom{2n+2l}{n}^{-1} \sum_{j=0}^n \binom{n}{j} U^{n-j} D^{n-j} \frac{(n+l)!}{(n+l-j)!} \binom{2n+2l-j}{n-j}.
\end{aligned}$$

Substituting (14) and (12) and replacing  $j$  by  $n-k$  we obtain

$$O(D^n U^n) = \binom{2n+2l}{n}^{-1} \frac{(n+l)!}{l!} \sum_{k=0}^n \binom{n}{k} \frac{(n+2l+1)_k}{(l+1)_k} \frac{(-1)^k}{2^k k!} \prod_{j=0}^{k-1} (-x/2 + 1/2 + j).$$

Since  $O(q^n p^n) = i^n O(D^n U^n)$  and  $T_1 = ix$ , we derive Eq. (8).

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